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A unified linear theory of homogeneous and stratified rotating fluids

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A unified picture of the linear dynamics of rotating fluids with given arbitrary stratification is presented. The range of stratification which lies outside the region of validity of both the theories of homogeneous fluids, $\sigma S < E^{\frac{2}{3}}$, and the strongly stratified fluids, $\sigma S > E^{\frac{1}{2}}$, is studied, where $\sigma S = \nu\alpha g \Delta T/\kappa\Omega^2 L$ and $E = \nu/\Omega L^2$. The transition from one dynamics to the other is elucidated by a detailed study of the intermediate region $E^{\frac{2}{3}} < \sigma S < E^{\frac{1}{2}}$. It is shown that, within this intermediate stratification range, the dynamics differs from that of either extreme case, except in the neighbourhood of horizontal boundaries where Ekman layers are present. In particular the side wall boundary layer exhibits a triple structure and is made up of (i) a buoyancy sublayer of thickness $(\sigma S)^{-\frac{1}{4}}E^{\frac{1}{2}}$ in which the viscous and buoyancy forces balance, (ii) an intermediate hydrostatic, baroclinic layer of thickness $(\sigma S)^{\frac{1}{2}}$ and (iii) an outer $E^{\frac{1}{4}}$-layer which is analogous to the one occurring in a homogeneous fluid. In the interior, the dynamics is mainly controlled by Ekman-layer suction, but displays hybrid features; in particular the dynamical fields can be decomposed into a 'homogeneous component' which satisfies the Taylor–Proudman theorem, and into a 'stratified component' which is baroclinic and which satisfies a thermal wind relation. In all regions the structure of the flow is displayed in detail.

1. Introduction

In a recent paper (Barcilon & Pedlosky (1967), hereafter referred to as B & P) we examined the linear theory for steady, rotating, stratified fluid motions. That analysis was pivoted, in a crucial way, on the assumption that the stratification was substantial, i.e. that the internal, rotational Froude number was order one. The results of the analysis could therefore not be directly applied to weakly or non-stratified rotating fluids. Consequently, it was not clear how the results would merge with those of the linear theory of *homogeneous*, rotating flows as the stratification is decreased.

In the present paper, we propose to examine how this transition occurs and to give a unified picture of the linear dynamics of rotating fluids which ties the results of B & P to those of the theory of homogeneous fluids. Because of the singular nature of the dynamics, manifested by the existence of spatial non-uniformities within the fluid (the various boundary layers), it is not possible to find useful, asymptotic solutions of the governing equations which are *uniformly valid* in the parametric measure of the stratification strength, S. As the homogeneous limit is approached for a fluid of small viscosity, the resulting dynamics will depend on the relation between S and the Ekman number E which is the parametric measure of the viscosity. That is, in the double limit $S \to 0$, $E \to 0$ it is necessary to specify a relation between E and S. In fact, three adjacent regions in parameter space emerge, namely (i) $\sigma S < E^{\frac{2}{3}}$, (ii) $E^{\frac{2}{3}} < \sigma S < E^{\frac{1}{2}}$, and (iii) $\sigma S > E^{\frac{1}{2}}$ (σ is the Prandtl number); and the character of the motions differs greatly in each region. In region (i), i.e. for stratifications smaller than $E^{\frac{2}{3}}$, we shall show that the fluid behaves essentially as if it were homogeneous; region (iii) corresponds to the case of strongly stratified fluid flows already examined in B & P, while, in the intermediate region (ii), the dynamics will have a hybrid nature, exhibiting features of both homogeneous and stratified fluids. Since the two extreme regions are well understood, in the present paper we shall focus our attention on the intermediate region $E^{\frac{2}{3}} < \sigma S < E^{\frac{1}{2}}$.

As in B & P, we shall restrict our attention to steady motions and to the case where the rotation $\mathbf{\Omega}$ and gravity $\mathbf{g}$ are antiparallel. Furthermore, we shall only consider axisymmetric flows within cylindrical containers of circular cross-section. These last restrictions are made for the sake of simplicity in presentation and could be relaxed without altering most of our results.

2. Formulation

The equations governing the steady motions of an incompressible, viscous, heat-conducting fluid, written in a co-ordinate frame rotating with angular velocity Ω about the vertical are

$$\mathbf{q}\,.\nabla\mathbf{q}+2\Omega\hat{k}\times\mathbf{q} = -\frac{1}{\rho}\nabla p - g\hat{k}+\frac{\Omega^2}{2}\nabla|\hat{k}\times\mathbf{r}|^2+\nu\nabla^2\mathbf{q},$$

$$\nabla.\mathbf{q} = 0,$$

$$\mathbf{q}\,.\nabla T = \kappa\nabla^2 T,$$

$$\rho = \rho_0[1-\alpha(T-T_0)],$$

where $\mathbf{q}$, p, ρ and T are respectively the velocity, pressure, density and temperature of the fluid at a point $\mathbf{r}$; ν and κ are the constant kinematic viscosity and thermometric conductivity, $\hat{k}$ is a unit vertical vector. The state equation is assumed to be a linear relation between T and ρ, where ρ_0 and T_0 are constant reference values of the density and temperature.

Assuming further, as in B & P, that the Froude number Ω^2L^2/g is small, the equilibrium density and temperature are linear functions of height, viz.

$$\rho_s = \rho_0[1-\alpha\Delta T(z/L)],$$

$$T_s = T_0+(\Delta T)z/L,$$

where L is the height of the container and ΔT is the basic vertical temperature difference ($\Delta T \geqslant 0$). Denoting by ϵ the departure from the state of rigid rotation and linear vertical stratification, we introduce dimensionless variables (denoted by asterisks) as follows:

$$\mathbf{q} = \epsilon\Omega L\mathbf{q}_*; \quad T = T_s+\frac{\epsilon\Omega^2L}{\alpha g}T_*; \quad r = L\mathbf{r}_*;$$

$$p = p_0-\rho_0 gLz_*-\tfrac{1}{2}\rho_0 gL\alpha\Delta T z_*^2+\epsilon\rho_0\Omega^2L^2p_*.$$

Assuming that the Rossby number ϵ is small, the dimensionless equations of motion are

$$2\hat{k}\times\mathbf{q} = -\nabla p + T\hat{k} + E\nabla^2\mathbf{q}, \tag{2.1}$$

$$\nabla.\mathbf{q} = 0, \tag{2.2}$$

$$\sigma S\hat{k}.\mathbf{q} = E\nabla^2 T, \tag{2.3}$$

where the asterisks have been dropped and $E = \nu/\Omega L^2$ and $\sigma = \nu/\kappa$ are respectively the Ekman and Prandtl numbers; while $S = \alpha\Delta T g/\Omega^2 L$ represents a measure of the stratification, and can be thought of as a rotational Richardson number. It should be noted that S and σ enter in the equations of motion only in the combination σS, which will be referred to as the stratification.

We shall assume that the flow is mechanically driven, say by a differential motion of the upper and/or lower boundaries, and that the heat flux through all the walls due to the motion is zero, i.e.

$$\hat{n}.\nabla T = 0 \tag{2.4}$$

on the boundaries. This boundary condition is necessary in order to include in our analysis the limiting case of a homogeneous fluid. Indeed, for $S = 0$, the heat equation (2.3) with (2.4) imply that T is a constant. Alternatively, we could replace the no-flux condition (2.4) by one requiring that T is a constant on the walls. The no-flux condition however simplifies the details of the results. The dynamical boundary conditions are of the form

$$\left.\begin{aligned} \mathbf{q} &= q_B(r)\hat{\theta}, &&\text{on}\quad z = 0;\\ \mathbf{q} &= q_T(r)\hat{\theta}, &&\text{on}\quad z = 1;\\ \mathbf{q} &= 0, &&\text{on}\quad r = a;\end{aligned}\right\} \tag{2.5}$$

where $\hat{\theta}$ is a unit circumferential vector and a is the dimensionless radius of the cylinder.

Although no formal asymptotic expansion in powers of E will be performed, we shall make use of boundary-layer techniques. As in the case of both homogeneous and strongly stratified rotating fluids, three distinct fluid regions exist: (i) Ekman layers along horizontal boundaries; (ii) side wall boundary layers; and (iii) an interior region. *Whenever they are present* the structure of the Ekman layer is independent of the size of the stratification, since the essential balance of forces, viz. Coriolis *vs.* viscous, involves horizontal motions which are unaffected by the stratification. We shall therefore not discuss the Ekman layers further and make use, when necessary, of the fundamental relation between the vorticity and the vertical velocity at the edge of the Ekman layer, i.e. that the Ekman layer suction is

$$w = \mp\tfrac{1}{2}E^{\frac{1}{2}}\hat{k}.\nabla\times(\mathbf{q}-\mathbf{q}_{\text{wall}})\quad\text{at}\quad z = \tfrac{1}{2}\pm\tfrac{1}{2}, \tag{2.6}$$

where

$$\mathbf{q}_{\text{wall}} = \begin{cases} q_T\hat{\theta}, & \text{on}\quad z = 1;\\ q_B\hat{\theta}, & \text{on}\quad z = 0.\end{cases}$$

One can now see that $\sigma S = E^{\frac{1}{2}}$ is a critical stratification. Two distinct mechanisms can control the size of the interior vertical velocity, the diffusion of heat and the Ekman-layer suction. From the heat equation we see that the interior vertical velocity is at most $O(E/\sigma S)$, while the vertical velocity pumped by the Ekman layers is $O(E^{\frac{1}{2}})$. Consequently, if $\sigma S > E^{\frac{1}{2}}$, the stratification will inhibit the Ekman-layer suction (and in fact will eliminate the $O(1)$ Ekman layers), as was the case in B & P. However, if $\sigma S < E^{\frac{1}{2}}$, the effects of the Ekman layers will dominate. Therefore the role of the Ekman layers (and their existence) will differ according to whether $\sigma S/E^{\frac{1}{2}} \gtrless 1$.

Postponing a discussion of the second critical stratification $\sigma S = O(E^{\frac{2}{3}})$ until we consider the side wall boundary layer, let us now turn to the dynamics of the interior region in the case $\sigma S < E^{\frac{1}{2}}$, which is the parameter region of interest here.

3. Interior region

To $O(E)$, the interior zonal flow is in geostrophic balance. As a result the pressure is $O(1)$ and the vertical momentum equation implies that the temperature is at most of $O(1)$. If T *were* $O(1)$, then the vertical velocity obtained from the heat equation would be $O(E/\sigma S)$, i.e. greater than $E^{\frac{1}{2}}$. However, if $w > O(E^{\frac{1}{2}})$ equation (2.6) would imply it vanishes on $z = 0, 1$ and, since w is z-independent according to the continuity equation ($u = O(E)$), w would be identically zero. We therefore infer that $w = O(E^{\frac{1}{2}})$ and consequently that T is $O(\sigma S/E^{\frac{1}{2}})$. We write for the interior

$$\left.\begin{aligned} u &= O(E), \\ v &= v_0 + \frac{\sigma S}{E^{\frac{1}{2}}} v_1 + \ldots, \\ w &= E^{\frac{1}{2}}\left\{w_0 + \frac{\sigma S}{E^{\frac{1}{2}}} w_1 + \ldots\right\}, \\ T &= \frac{\sigma S}{E^{\frac{1}{2}}}\{T_0 + \ldots\}, \\ p &= p_0 + \frac{\sigma S}{E^{\frac{1}{2}}} p_1 + \ldots. \end{aligned}\right\} \tag{3.1}$$

The $O(1)$ and $O(\sigma S/E^{\frac{1}{2}})$ interior equations are

$$\left.\begin{aligned} 2v_0 &= p_{0r}, & 0 &= p_{0z}, \\ w_{0z} &= 0, & w_0 &= \nabla^2 T_0, \end{aligned}\right\} \tag{3.2}$$

and

$$\left.\begin{aligned} 2v_1 &= p_{1r}, & 0 &= -p_{1z} + T_0 \\ w_{1z} &= 0, & w_1 &= \nabla^2 T_1. \end{aligned}\right\} \tag{3.3}$$

To lowest order, the interior dynamics is identical to that of a homogeneous fluid. As soon as $\sigma S < E^{\frac{1}{2}}$ the dominant part of the interior motion becomes two-dimensional, with p_0 and hence v_0 independent of z; the Taylor–Proudman theorem is satisfied and the interior $O(1)$ dynamics is consequently completely controlled by the Ekman-layer suction.

The $O(\sigma S/E^{\frac{1}{2}})$ flow is, however, affected by the temperature deviations produced by the $O(E^{\frac{1}{2}})$ Ekman-layer suction. In terms of p_1 the $O(\sigma S/E^{\frac{1}{2}})$ equation is

$$\nabla^2 p_{1zz} = 0, \tag{3.4}$$

which could have been deduced from the fundamental interior equation (3.13)′ of B & P by formally setting $\sigma S = 0$. The solution for the $O(\sigma S/E^{\frac{1}{2}})$ variables can be expressed as follows:

$$\left.\begin{aligned} T_0 &= \int_0^r \frac{dr'}{r'} \int_0^{r'} r'' w_0(r'')\,dr'' + \chi_z(r,z), \\ p_1 &= (z-\tfrac{1}{2}) \int_0^r \frac{dr'}{r'} \int_0^{r'} r'' w_0(r'')\,dr'' + \chi, \\ v_1 &= \frac{1}{2r}(z-\tfrac{1}{2}) \int_0^r r' w_0(r')\,dr' + \tfrac{1}{2}\chi_r, \end{aligned}\right\} \tag{3.5}$$

where χ is a harmonic function to be determined. Defining ϕ_T and ϕ_B such that

$$\mathbf{q}_{T,B} = \tfrac{1}{2}\hat{k} \times \nabla\phi_{T,B},$$

we can write the Ekman suction conditions (2.6) as

$$w_0 + \frac{\sigma S}{E^{\frac{1}{2}}} w_1 = -\frac{1}{2}\left[\tfrac{1}{2}\nabla_1^2(p_0 - \phi_T) + \frac{\sigma S}{E^{\frac{1}{2}}}\{\tfrac{1}{4}w_0 + \tfrac{1}{2}\nabla_1^2\chi\}\right] \quad \text{on} \quad z = 1, \tag{3.6a}$$

$$w_0 + \frac{\sigma S}{E^{\frac{1}{2}}} w_1 = +\frac{1}{2}\left[\tfrac{1}{2}\nabla_1^2(p_0 - \phi_B) + \frac{\sigma S}{E^{\frac{1}{2}}}\{-\tfrac{1}{4}w_0 + \tfrac{1}{2}\nabla_1^2\chi\}\right] \quad \text{on} \quad z = 0, \tag{3.6b}$$

where
$$\nabla_1^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}.$$

Since $\chi_{zz} = 0$ on $z = 0, 1$ from (2.4), we see that $\nabla_1^2\chi = 0$ on $z = 0, 1$ for χ is a harmonic function and that in turn

$$\chi = \text{const.} \quad \text{on} \quad z = 0, 1. \tag{3.7}$$

Therefore

$$\left.\begin{aligned} w_0 &= \tfrac{1}{8}\nabla_1^2(\phi_T - \phi_B), \\ w_1 &= -\tfrac{1}{64}\nabla_1^2(\phi_T - \phi_B), \\ p_0 &= \tfrac{1}{2}(\phi_T + \phi_B). \end{aligned}\right\} \tag{3.8}$$

Making use of (3.8) and (3.5) we find that

$$p = \tfrac{1}{2}(\phi_T + \phi_B) + \frac{\sigma S}{E^{\frac{1}{2}}}\left\{\frac{z-\frac{1}{2}}{8}(\phi_T - \phi_B) + \chi\right\}, \tag{3.9}$$

$$v = \frac{1}{4}\frac{d}{dr}(\phi_T + \phi_B) + \frac{\sigma S}{E^{\frac{1}{2}}}\left\{\frac{z-\frac{1}{2}}{16}\frac{d}{dr}(\phi_T - \phi_B) + \tfrac{1}{2}\chi_r\right\}, \tag{3.10}$$

which displays the hybrid nature of the flow. Indeed (3.10) suggests that we look upon v as being made up of an $O(1)$ 'homogeneous' component, which is z-independent and equal to the local average of the zonal velocities of the upper and lower boundaries, and of a 'stratified' or 'baroclinic' component.

The various 'stratified components' are only known to within a harmonic function χ which satisfies the boundary conditions (3.7). In order to complete the interior problem by specifying a boundary condition for χ at $r = a$, we must examine the side wall boundary layer.

4. Side wall boundary layer

Before we derive the equations valid inside the side wall boundary layer we can anticipate some of the results by means of heuristic arguments and of our knowledge of the dynamics of the extreme cases for the homogeneous fluids (Stewartson layers) and for the strongly stratified fluids (buoyancy layer).

In both extremes the zonal velocity remained in geostrophic balance:

$$2v = p_r, \tag{4.1}$$

while in the zonal momentum equation Coriolis and viscous forces balance:

$$2u = Ev_{rr}. \tag{4.2}$$

In the vertical equation of motion the pressure gradient, the buoyancy and the viscous forces might be comparable. Including all such effects *a priori*, we get

$$0 = -p_z + T + Ew_{rr}. \tag{4.3}$$

Finally the continuity and heat equations should be

$$u_r + w_z = 0, \tag{4.4}$$

$$\sigma S w = ET_{rr}. \tag{4.5}$$

We can postpone scaling the dependent variables by working with a single equation for a single field, say v, namely

$$E^2 \frac{\partial^6 v}{\partial r^6} + \sigma S \frac{\partial^2 v}{\partial r^2} + 4 \frac{\partial^2 v}{\partial z^2} = 0, \tag{4.6}$$

which is a generalization of the equations for the Stewartson and buoyancy layers. Indeed, whenever the first and third terms in (4.6) are comparable we recover the equation for the Stewartson $E^{\frac{1}{3}}$-layer. This balance is possible only for stratification, $\sigma S < E^{\frac{2}{3}}$, which emerges as the second critical stratification. For $\sigma S > E^{\frac{2}{3}}$, the second term in (4.6) is no longer negligible and we anticipate a balance between the first and second terms in (4.6) yielding a boundary layer whose thickness is $O((\sigma S)^{-\frac{1}{4}} E^{\frac{1}{2}})$, i.e. thicker than $E^{\frac{1}{2}}$. This observation is important since we are now entitled to use the Ekman-layer suction conditions (2.6) at the top and bottom boundaries *within* the side wall boundary layers. Making the usual boundary-layer approximations, these conditions can be written in terms of v alone as

$$4E^{\frac{1}{2}} v_z \pm \left(E^2 \frac{\partial^4 v}{\partial r^4} + \sigma S v \right) = 0 \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{4.7}$$

We can now look for solutions of (4.6)–(4.7) of the form

$$v = f(z) \exp\{-\beta(a-r)\}.$$

Equation (4.7) will then provide a pair of characteristic equations for β and hence will yield the possible boundary-layer thicknesses which will be $O(\beta^{-1})$. These characteristic equations are

$$(\sigma S + E^2\beta^4)^{\frac{1}{2}} [2\beta E^{\frac{1}{2}} + (\sigma S + E^2\beta^4)^{\frac{1}{2}} \tan \tfrac{1}{4}\beta(\sigma S + E^2\beta^4)^{\frac{1}{2}}] = 0, \tag{4.8a}$$

$$(\sigma S + E^2\beta^4)^{\frac{1}{2}} [-2\beta E^{\frac{1}{2}} \tan \tfrac{1}{4}\beta(\sigma S + E^2\beta^4)^{\frac{1}{2}} + (\sigma S + E^2\beta^4)^{\frac{1}{2}}] = 0. \tag{4.8b}$$

Within the range of stratification under consideration, $E^{\frac{2}{3}} < \sigma S < E^{\frac{1}{2}}$, it is easy to verify that the side wall boundary layer will have a *triple scale structure* with sublayers of thickness $E^{\frac{1}{4}}$, $(\sigma S)^{\frac{1}{2}}$ and $(\sigma S)^{-\frac{1}{4}}E^{\frac{1}{2}}$.

Making use of this information, let us go back to the equations of motion and investigate each of these sublayers separately. Note that, since the interior velocity is composed of a 'homogeneous component' of $O(1)$ and a baroclinic or 'stratified' component of $O(\sigma S/E^{\frac{1}{2}})$ and since both components must be adjusted to zero on the side wall by the boundary layers, we must obtain the boundary-layer fields correctly to $O(\sigma S/E^{\frac{1}{2}})$.

The $E^{\frac{1}{4}}$ layer

Starting with the thickest layer, let us introduce a stretched variable ξ defined as

$$\xi = (a-r)\,E^{-\frac{1}{4}}\left(1+\lambda_1\frac{\sigma S}{E^{\frac{1}{2}}}+\ldots\right), \tag{4.9a}$$

and let us scale the various fields which are *corrections* to the interior fields as

$$\left.\begin{aligned} u &= E^{\frac{1}{2}}\left\{\bar{u}+\frac{\sigma S}{E^{\frac{1}{2}}}\bar{u}_1+\ldots\right\},\\ v &= \left\{\bar{v}+\frac{\sigma S}{E^{\frac{1}{2}}}\bar{v}_1+\ldots\right\},\\ w &= E^{\frac{1}{4}}\left\{\bar{w}+\frac{\sigma S}{E^{\frac{1}{2}}}\bar{w}_1+\ldots\right\},\\ p &= E^{\frac{1}{4}}\left\{\bar{p}+\frac{\sigma S}{E^{\frac{1}{2}}}\bar{p}_1+\ldots\right\},\\ T &= \frac{\sigma S}{E^{\frac{1}{4}}}\left\{\bar{T}+\frac{\sigma S}{E^{\frac{1}{2}}}\bar{T}_1+\ldots\right\}, \end{aligned}\right\} \tag{4.9b}$$

where the constant λ_1, introduced into the co-ordinate stretching (4.9*a*) will be determined by ensuring that the boundary-layer expansions (4.9*b*) are uniformly valid in ξ to order $\sigma S/E^{\frac{1}{2}}$.

To the lowest order the equations of motion become

$$\left.\begin{aligned} 2\bar{v} &= -\bar{p}_\xi, & 2\bar{u} &= \bar{v}_{\xi\xi},\\ 0 &= -\bar{p}_z, & \bar{u}_\xi &= \bar{w}_z; \end{aligned}\right\} \tag{4.10}$$

i.e. they are identical to those describing the outer Stewartson layer for homogeneous fluids; while the induced temperature is computed from the heat equation,

$$\bar{T}_{\xi\xi} = \bar{w}. \tag{4.11}$$

As in the homogeneous case, $\bar{p}$ is independent of z and hence both $\bar{u}$ and $\bar{v}$ are z-independent. However, in the next-order correction, the pressure gradient $\bar{p}_{1z}$ is balanced by the buoyancy of the lower-order temperature, viz.

$$\bar{p}_{1z} = \bar{T},$$

whereas in the homogeneous case it would be balanced by the viscous stresses and be of higher order. As a result the analogy with the homogeneous $E^{\frac{1}{4}}$-layer is valid only to the lowest order. Omitting certain calculations we derive the following expressions for the various $E^{\frac{1}{4}}$-layer fields correct to $O(\sigma^2S^2/E)$:

$$\left.\begin{aligned} v &= \left[M+\frac{\sigma S}{E^{\frac{1}{2}}}\left\{-\frac{M}{4}(z-\tfrac{1}{2})^2+\frac{M}{24}+N\right\}\right]\exp\left\{-\sqrt{2}\left(1-\frac{\sigma S}{48E^{\frac{1}{2}}}\right)\xi\right\},\\ u &= E^{\frac{1}{4}}\left[M+\frac{\sigma S}{E^{\frac{1}{2}}}\left\{-\frac{M}{4}(z-\tfrac{1}{2})^2+N\right\}\right]\exp\left\{-\sqrt{2}\left(1-\frac{\sigma S}{48E^{\frac{1}{2}}}\right)\xi\right\},\\ w &= -\sqrt{2}E^{\frac{1}{4}}(z-\tfrac{1}{2})\left[M+\frac{\sigma S}{E^{\frac{1}{2}}}\left\{-\frac{M}{12}(z-\tfrac{1}{2})^2+N-\frac{M}{48}\right\}\right]\exp\left\{-\sqrt{2}\left(1-\frac{\sigma S}{48E^{\frac{1}{2}}}\right)\xi\right\},\\ T &= -\frac{1}{\sqrt{2}}\frac{\sigma S}{E^{\frac{1}{4}}}\left[M+\frac{\sigma S}{E^{\frac{1}{2}}}\left\{-\frac{M}{12}(z-\tfrac{1}{2})^2+N+\frac{M}{48}\right\}\right]\exp\left\{-\sqrt{2}\left(1-\frac{\sigma S}{48E^{\frac{1}{2}}}\right)\xi\right\}, \end{aligned}\right\} \tag{4.12}$$

where M and N are arbitrary constants.

Hydrostatic layer

The next thickest layer, the hydrostatic layer with thickness $(\sigma S)^{\frac{1}{2}}$, shares many of the physical features of the $E^{\frac{1}{4}}$-layer. The vertical momentum equation is still hydrostatic, but the buoyancy force for this layer is as large as the vertical pressure gradient. The layer can be extracted by balancing the last two terms in (4.6). In fact, if the last two terms in (4.6) are balanced, and correspondingly the second term in (4.7) is dropped, the resulting simplified eigenvalue equation yields both the $E^{\frac{1}{4}}$-layer, as the first mode, and the hydrostatic layer as all the higher modes.

Using a caret to denote the various *correction* fields within this boundary layer and defining a stretched variable

$$\eta = (\sigma S)^{-\frac{1}{2}}(a-r),$$

the boundary-layer equations are

$$\left.\begin{aligned} -2\hat{v} &= \hat{p}_\eta, \quad 2\hat{u} = \hat{v}_{\eta\eta},\\ 0 &= -\hat{p}_z+\hat{T}, \quad \hat{u}_\eta = \hat{w}_z, \quad \hat{w} = \hat{T}_{\eta\eta}, \end{aligned}\right\} \tag{4.13}$$

where the *correction* fields have been scaled in the following way:

$$\left.\begin{aligned} u &= E^{\frac{1}{2}}\hat{u}, \quad v = \sigma S E^{-\frac{1}{2}}\hat{v}, \quad w = (\sigma S)^{-\frac{1}{2}}E^{\frac{1}{2}}\hat{w},\\ T &= (\sigma S)^{\frac{1}{2}}E^{-\frac{1}{2}}\hat{T}, \quad P = (\sigma S)^{\frac{3}{2}}E^{-\frac{1}{2}}\hat{p}. \end{aligned}\right\} \tag{4.14}$$

The Ekman suction condition (4.7) reduces to $\hat{w} = 0$ on $z = 0, 1$ so that a convenient representation of the caret fields are:

$$\left.\begin{aligned} \hat{u} &= -\frac{1}{2}\sum_{n=1}^{\infty} A_n \cos n\pi z \exp(-2n\pi\eta),\\ \hat{v} &= -\frac{1}{4}\sum_{n=1}^{\infty} \frac{A_n}{(n\pi)^2}\cos n\pi z \exp(-2n\pi\eta),\\ \hat{w} &= \sum_{n=1}^{\infty} A_n \sin n\pi z \exp(-2n\pi\eta),\\ \hat{T} &= \frac{1}{4}\sum_{n=1}^{\infty}(n\pi)^{-2}A_n \sin n\pi z \exp(-2n\pi\eta). \end{aligned}\right\} \tag{4.15}$$

These expressions for the $(\sigma S)^{\frac{1}{2}}$-layer fields are reminiscent of those for the $E^{\frac{1}{3}}$-layer on account of their vertical structure. It is interesting to observe that, as σS approaches the critical stratification $E^{\frac{2}{3}}$, the thickness of the $(\sigma S)^{\frac{1}{2}}$-layer approaches $E^{\frac{1}{3}}$. However, the fact that this layer is hydrostatic implies that the dynamics of the $E^{\frac{1}{3}}$-layer cannot be completely represented by *only* the limiting dynamics of the $(\sigma S)^{\frac{1}{2}}$-layer as $\sigma S \to E^{\frac{2}{3}}$.

Buoyancy layer

The most narrow layer is the buoyancy layer. The layer is so thin that the viscous stresses in the vertical momentum equation become as large as the buoyancy forces while the vertical pressure gradient becomes negligible. The thickness of the layer is $O(E^{\frac{1}{2}}(\sigma S)^{-\frac{1}{4}})$ and we showed in B & P that this was the *only* side wall boundary layer when $\sigma S = O(1)$, i.e. for strong stratification. Introducing the appropriate stretched variable

$$\rho = (\sigma S)^{\frac{1}{4}} E^{-\frac{1}{2}}(a-r),$$

and denoting the correction fields in the buoyancy layer with a tilde we write

$$\left.\begin{aligned} u = E^{\frac{1}{2}}\tilde{u}, \quad v = (\sigma S)^{-\frac{1}{4}} E^{\frac{1}{2}}\tilde{v}, \\ w = (\sigma S)^{\frac{1}{4}}\tilde{w}, \quad T = (\sigma S)^{\frac{3}{4}}\tilde{T}, \quad p = (\sigma S)^{-\frac{3}{4}} E\tilde{p}, \end{aligned}\right\} \tag{4.16}$$

yielding as the governing equations for the buoyancy layer

$$\left.\begin{aligned} -2\tilde{v} = \tilde{p}_\rho, \quad 2\tilde{u} = \tilde{v}_{\rho\rho}, \\ 0 = \tilde{T} + \tilde{w}_{\rho\rho}, \quad \tilde{u}_\rho = \tilde{w}_z, \quad \tilde{w} = \tilde{T}_{\rho\rho}. \end{aligned}\right\} \tag{4.17}$$

Since each of the three sublayers can carry as vertical mass flux of $O(E^{\frac{1}{2}})$, it therefore follows that the vertical velocity in the buoyancy layer (which is the thinnest) will be the largest and hence must vanish at the wall. As a result we deduce from (4.17) that:

$$\left.\begin{aligned} \tilde{u} &= -\frac{dA}{dz} e^{-\rho/\sqrt{2}} \sin\left(\frac{\rho}{\sqrt{2}} + \frac{\pi}{4}\right), \\ \tilde{v} &= -2\frac{dA}{dz} e^{-\rho/\sqrt{2}} \cos\left(\frac{\rho}{\sqrt{2}} + \frac{\pi}{4}\right), \\ \tilde{w} &= A(z) e^{-\rho/\sqrt{2}} \sin \rho/\sqrt{2}, \\ \tilde{T} &= A(z) e^{-\rho/\sqrt{2}} \cos \rho/\sqrt{2}. \end{aligned}\right\} \tag{4.18}$$

It is worth while noting that, as σS approaches $E^{\frac{2}{3}}$, the second critical stratification, the non-hydrostatic buoyancy layer widens into a layer of thickness $E^{\frac{1}{3}}$, merging with the $(\sigma S)^{\frac{1}{2}}$-layer, becoming the non-hydrostatic Stewartson $E^{\frac{1}{3}}$-layer.

We are now in a position to correct the interior fields so as to satisfy the various boundary conditions at the side wall and, hence, to obtain a boundary condition for the as yet undetermined harmonic function χ.

5. Closure of the interior problem

We have already explicitly satisfied the boundary condition for w by means of the buoyancy layer so that we now only must satisfy the boundary conditions for u, v and T. The radial velocities in all three sublayers are $O(E^{\frac{1}{2}})$ and, since u in the interior is $O(E)$, we must have

$$\bar{u}+\hat{u}+\tilde{u} = 0 \quad \text{at} \quad r = a. \tag{5.1}$$

Instead of using (5.1) we shall use the equivalent condition that the outer wall be a streamline for the meridional motion. Since the motion is axisymmetric, this in turn is equivalent to balancing the vertical mass fluxes of the interior and boundary layers. Integrating the continuity equation for each layer across its width yields

$$(\bar{u}+\hat{u}+\tilde{u})_{r=a} = -\frac{d}{dz}\left\{\int_0^\infty \bar{w}\,d\xi+\int_0^\infty \hat{w}\,d\eta+\int_0^\infty \bar{w}\,d\rho\right\}$$

and using (5.1) we obtain

$$\int_0^\infty \bar{w}\,d\xi+\int_0^\infty \hat{w}\,d\eta+\int_0^\infty \tilde{w}\,d\rho = \text{const.} = Q. \tag{5.2}$$

The constant Q is easily evaluated by noting that $2\pi a E^{\frac{1}{2}}Q$ represents the vertical mass transport via the side wall boundary layer, which must be equal and opposite to the vertical transport via the interior, i.e.

$$2\pi a Q = -2\pi\int_0^a r w_0\,dr.$$

Using the expression for w_0 given in (3.8) we see that

$$Q = -\tfrac{1}{8}(d/dr)(\phi_T-\phi_B)_{r=a} = -\tfrac{1}{4}[v_T(a)-v_B(a)]. \tag{5.3}$$

Turning now to the heat-flux condition and noting that the temperature gradient within each sublayer is $O(\sigma S/E^{\frac{1}{2}})$ we must require that

$$T_{0r}-\bar{T}_\xi-T_\eta-T_\rho = 0 \quad \text{at} \quad r = a. \tag{5.4}$$

However, by integrating the heat equation across each sub-layer we deduce

$$\int_0^\infty \bar{w}\,d\xi+\int_0^\infty \hat{w}\,d\eta+\int_0^\infty \tilde{w}\,d\rho = -[\bar{T}_\xi+\hat{T}_\eta+\tilde{T}_\rho]_{r=a},$$

or, using (5.2), (5.3) and (5.4),

$$T_{0r} = \tfrac{1}{8}([d/dr](\phi_T-\phi_B))_{r=a} = \tfrac{1}{4}(v_T(a)-v_B(a)), \tag{5.5a}$$

$$\bar{T}_\xi+\hat{T}_\eta+\tilde{T}_\rho = \tfrac{1}{4}(v_T(a)-v_B(a)). \tag{5.5b}$$

We have therefore succeeded in deriving a boundary condition for the interior temperature T_0. According to (3.5) and (3.8),

$$T_{0r} = \tfrac{1}{8}[d/dr](\phi_T-\phi_B)+\chi_{rz} \tag{5.6}$$

and therefore, using (5.5a),

$$\chi_{rz} = 0, \quad \text{on} \quad r = a. \tag{5.7}$$

Since the harmonic function χ is equal to a constant on $z = 0, 1$ we deduce without loss of generality that χ itself is zero. Consequently the interior fields are

$$\left.\begin{aligned}
u &= \frac{E}{4}\left\{\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left[(v_T(r)+v_B(r))+\frac{1}{4}\frac{\sigma S}{E^{\frac{1}{2}}}(z-\tfrac{1}{2})(v_T(r)-v_B(r))\right]\right\},\\
v &= \tfrac{1}{2}[v_T(r)+v_B(r)]+\frac{\sigma S}{8E^{\frac{1}{2}}}(z-\tfrac{1}{2})[v_T(r)-v_B(r)]+O\left(\frac{\sigma S}{E^{\frac{1}{2}}}\right)^2,\\
w &= \frac{E^{\frac{1}{2}}}{4}\left(1-\frac{\sigma S}{8E^{\frac{1}{2}}}\right)\frac{1}{r}\frac{d}{dr}r[v_T(r)-v_B(r)]+O\left(\frac{\sigma S}{E^{\frac{1}{2}}}\right)^2,\\
T &= \frac{1}{4}\frac{\sigma S}{E^{\frac{1}{2}}}\int_0^r[v_T(r')-v_B(r')]\,dr'+O\left(\frac{\sigma S}{E^{\frac{1}{2}}}\right)^2,\\
p &= \tfrac{1}{2}(\phi_T+\phi_B)+\frac{1}{8}\frac{\sigma S}{E^{\frac{1}{2}}}(z-\tfrac{1}{2})(\phi_T-\phi_B)+O\left(\frac{\sigma S}{E^{\frac{1}{2}}}\right)^2.
\end{aligned}\right\} \quad (5.8)$$

The expressions (5.8) constitute a good representation for the interior fields for stratifications σS smaller than $E^{\frac{1}{2}}$ and in particular they are valid in the limit $\sigma S \to 0$, i.e. for homogeneous fluids. In fact $\sigma S = E^{\frac{2}{3}}$ is a critical stratification only for the side wall boundary layer and not for the interior flow, which, for $\sigma S < E^{\frac{1}{2}}$, is primarily controlled by the Ekman layers.†

In order to complete the entire solution we must determine the coefficients M, N, A_n, as well as the function $A(z)$ which enter the expressions for the side wall sublayers.

The boundary conditions for the $O(1)$ and $O(\sigma S E^{-\frac{1}{2}})$ zonal flows are:

$$v_0+\bar{v} = 0 \quad \text{on} \quad r = a; \quad (5.9a)$$

$$v_1+\bar{v}_1+\hat{v} = 0 \quad \text{on} \quad r = a. \quad (5.9b)$$

Making use of (4.12), (4.15) and (5.8), we find that (5.9a,b) imply

$$M = -\tfrac{1}{2}(v_T(a)+v_B(a)) \quad (5.10)$$

and

$$\tfrac{1}{8}(z-\tfrac{1}{2})[v_T(a)-v_B(a)]+\tfrac{1}{8}\{(z-\tfrac{1}{2})^2-\tfrac{1}{6}\}[v_T(a)+v_B(a)] + N - \frac{1}{4}\sum_{n=1}^{\infty}(n\pi)^{-2}A_n\cos n\pi z = 0. \quad (5.11)$$

Consequently

$$N = \tfrac{1}{96}(v_T(a)+v_B(a)), \quad (5.12)$$

and

$$A_n = 2[(-1)^n v_T(a)+v_B(a)], \quad (5.13)$$

† A representation of v, w and T uniformly valid over the entire stratification range $0 \leqslant \sigma S < 1$ in the inviscid interior of the fluid, which reduces to (5.8) when $\sigma S E^{-\frac{1}{2}} \ll 1$, can be shown to be

$$v = \tfrac{1}{2}[v_T(r)+v_B(r)]+\tfrac{1}{2}[v_T(r)-v_B(r)]\frac{\sigma S}{4E^{\frac{1}{2}}(1+\sigma S/8E^{\frac{1}{2}})}(z-\tfrac{1}{2}),$$

$$w = \frac{E^{\frac{1}{2}}}{4(1+\sigma S/8E^{\frac{1}{2}})}\frac{1}{r}\frac{d}{dr}[r(v_T(r)-v_B(r))],$$

$$T = \frac{\sigma S}{4E^{\frac{1}{2}}(1+\sigma S/8E^{\frac{1}{2}})}\int_0^r[v_T(r')-v_B(r')]\,dr'.$$

Note that as $\sigma S/E^{\frac{1}{2}}$ becomes large, w falls to $O(E)$ (no Ekman suction) while $v(r,z)$ increases linearly from the plate velocity at the bottom to the plate velocity at the top, as predicted in B & P.

which completes the specification of the $(\sigma S)^{\frac{1}{2}}$-layer and the $E^{\frac{1}{4}}$-layer. Finally (5.5*b*), which can be written

$$-\tfrac{1}{2}(z-\tfrac{1}{2})\,[v_T(a)+v_B(a)]-\frac{1}{2}\sum_{n=1}^{\infty}(n\pi)^{-1}[(-1)^n v_T(a)+v_B(a)]\sin n\pi z$$
$$-\frac{A(z)}{2}=\tfrac{1}{4}(v_T(a)-v_B(a)), \quad (5.14)$$

together with the following Fourier series

$$z-\tfrac{1}{2}=-\sum_{n=1}^{\infty}\frac{1}{n\pi}[(-1)^n+1]\sin n\pi z,$$

$$1=2\Sigma\frac{1}{n\pi}[(-1)^n-1]\sin n\pi z,$$

implies that
$$A(z)=0. \quad (5.15)$$

So that, *to this order*, the buoyancy layer is absent. This is reminiscent of the disappearance of the buoyancy layer in the strongly stratified regime ($\sigma S = O(1)$) for the case of insulated side walls considered by B & P. Nevertheless, the buoyancy layer will *in general* be present, and of course, even here, it will be present to higher order.

6. Conclusion

The present analysis, together with that of B & P, enables us now to present a unified picture of the linear dynamics of stratified rotating fluids.

Two critical stratifications of $O(E^{\frac{2}{3}})$ and $O(E^{\frac{1}{2}})$ arise and divide the $(E, \sigma S)$-space into three distinct regions. For $\sigma S < E^{\frac{2}{3}}$, the fluid, in *all* regions, acts as if it were homogeneous. In particular: (i) its interior dynamics is strongly controlled by the Ekman-layer suction, (ii) Stewartson boundary layers of thicknesses $E^{\frac{1}{3}}$, $E^{\frac{1}{4}}$ are needed on the vertical walls, and (iii) the Taylor–Proudman theorem is valid in the interior.

As σS increases beyond $E^{\frac{2}{3}}$, the effects of the stratification are felt most strongly within the Stewartson $E^{\frac{1}{3}}$-layer and more specifically in the vertical momentum balance. When $\sigma S > E^{\frac{2}{3}}$ the buoyancy force is no longer negligible. This additional force upsets the balance between the vertical pressure gradient and the viscous force, and splits the $E^{\frac{1}{3}}$-layer into two layers: a thinner buoyancy layer in which the viscous stresses balance the buoyancy and a thicker hydrostatic layer where the buoyancy balances the vertical pressure gradient. The $E^{\frac{1}{4}}$-layer is not as drastically modified, but some degree of baroclinicity affects its structure. The interior dynamics is primarily controlled by the Ekman layers whose existence is not affected as σS increases beyond $E^{\frac{2}{3}}$. As a result the $\sigma S \sim E^{\frac{2}{3}}$ transition is a smooth one as far as the interior fields are concerned. However, except to lowest order, the Taylor–Proudman theorem is no longer valid and the zonal velocity has a baroclinic 'thermal wind' component in addition to its 'homogeneous', z-independent component. These two parts become comparable as $\sigma S \sim E^{\frac{1}{2}}$. When σS is greater than $E^{\frac{1}{2}}$ the stratification is sufficiently strong to inhibit the Ekman-layer suction. This is essentially the parameter space region investigated in B & P.

As the Ekman layers disappear, the hydrostatic $(\sigma S)^{\frac{1}{2}}$-layer, which has emerged with the $E^{\frac{1}{2}}$-layer for $\sigma S \sim E^{\frac{1}{2}}$, penetrates into the interior of the fluid, which then becomes controlled by viscous–diffusive processes.

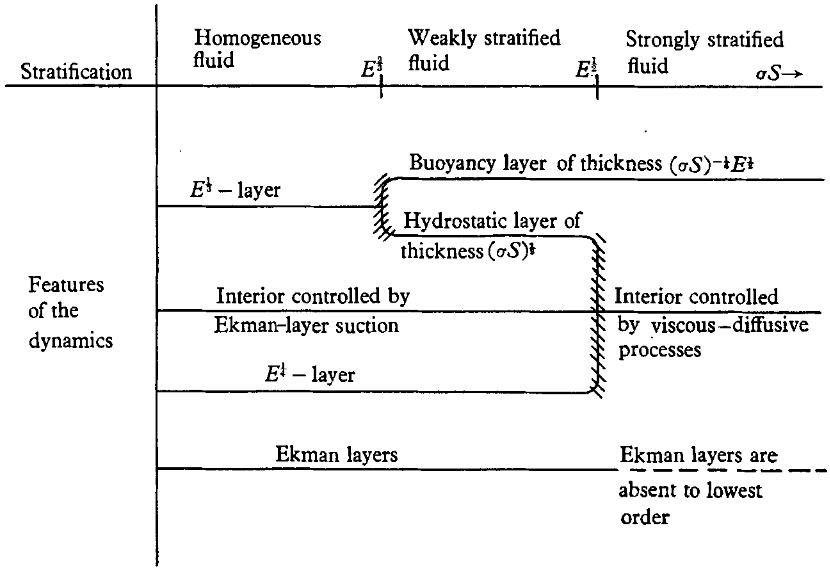

FIGURE 1. Schematic description of the elements of the dynamics of rotating fluids for various stratifications.

The above conclusions are summarized in figure 1, in which the modifications of the various features of the dynamics of stratified rotating fluids are schematically represented as the stratification σS is increased.

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